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A CLASS OF SINGULARLY PERTURBED BOUNDARY VALUE PROBLEMS IN A TWO-DIMENSIONAL ELASTICITY THEORY

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In studying boundary value problems for an elastic medium reinforced by a family of very stiff fibers, authors are frequently concerned with the limiting case model of an elastic medium with inextensible fibers, i.e., deformation along a specified direction is equal to zero [1]. With the exception of [2], little attention has been devoted to ascertaining the correctness of this limiting model; in [2], under rather stringent assumptions concerning the boundary of the domain, consideration was given to a boundary value problem with a specified stress vector on the boundary for a medium inextensible in a given direction and with direct reinforcement.

Here we prove a series of theorems concerned with the convergence of singularly perturbed problems of a given class to limiting solutions in corresponding Hilbert spaces; we also show that the limiting system of equations cannot coincide with the system resulting from the assumption of inextensibility. A concrete example of a similar situation appears in [3].

1. In a curvilinear orthogonal coordinate system in the plane, (α_1, α_2), we take the generalized Hooke's Law for an orthotropic material in the form [4]

$$\sigma_{11} = \varepsilon^{-2}e_{11} + b_{12}e_{22},\ \sigma_{22} = b_{12}e_{11} + b_{22}e_{22},\ \sigma_{12} = 2e_{12}, \tag{1.1}$$

where the dimensionless stresses are taken with reference to the shear modulus G; the axes of orthotropicity coincide with the (α_1, α_2) axes; $\varepsilon^{-2} = b_{11}G^{-1} \gg 1$; $\varepsilon \ll 1$; $b_{22} - \varepsilon^2 b_{12}{}^2 > 0$; $b_{22} > 0$. Deformations may be represented in terms of displacements $u = (u_1, u_2)$ in the following way:

$$e_{11} = \frac{1}{h_1}\frac{\partial u_1}{\partial \alpha_1} + \frac{1}{h_1 h_2}\frac{\partial h_1}{\partial \alpha_2}u_2,$$

$$e_{22} = \frac{1}{h_2}\frac{\partial u_2}{\partial \alpha_2} + \frac{1}{h_1 h_2}\frac{\partial h_2}{\partial \alpha_1}u_1,$$

$$2e_{12} = \frac{h_2}{h_1}\frac{\partial}{\partial \alpha_1}\left(\frac{u_2}{h_2}\right) + \frac{h_1}{h_2}\frac{\partial}{\partial \alpha_2}\left(\frac{u_1}{h_1}\right),$$

h_1 and h_2 are the Lamé parameters ($h_1, h_2 \geq \text{const} > 0$, $h_{k.\alpha_j}$ (k, j = 1.2)), measurable and bounded in a compact simply connected domain Q with a piecewise-smooth boundary γ. We introduce the Hilbert space V of functions $v = (v_1, v_2)$, $v_k \in L^2(Q)$ (k = 1.2) with the finite norm

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$$\|v\|_V = \left\{\int_Q \left[\sum_{i,j=1}^{2} \left(\frac{\partial v_i}{\partial \alpha_j}\right)^2 + v_1^2 + v_2^2 \right] dQ \right\}^{1/2},$$

$$dQ = h_1 h_2 d\alpha_1 d\alpha_2$$

and the scalar product corresponding to this norm.

2. We consider a mixed problem of elasticity theory with given volume forces and a given stress vector on a portion γ_2 of the boundary. We assume that γ is representable in the form of the union $\gamma = \gamma_1 \cup \gamma_2$, where γ_2 consists of connected components of the boundary with equation $\alpha_2 \equiv$ const, the linear measure of γ_1 is different from zero, and $\gamma_1 \cap \gamma_2 = \Phi$. We note that an inequality of Korn's type [5] is valid in V: There exists a positive constant $c < 0$ such that

$$\|v\|^2_{[H^1(Q)]^2} \leqslant c \left(\sum_{i,j=1}^{2} \|e_{ij}(v)\|^2_{[L^2(Q)]} + \|v\|^2_{[L^2(Q)]^2} \right).$$

We shall present a variational formulation of the problem without writing down explicitly the complicated set of equations of elasticity theory. Let V_0 be a closed subspace of V characterized by the condition $v|_{\gamma_1} = 0$. We need to determine a two-component function $u^3 = (u_1^\varepsilon, u_2^\varepsilon) \in V_0$, such that for every $v \in V_0$ we have the integral identity

$$\int_Q \sigma_{ij}(u^\varepsilon) e_{ij}(v)\, dQ = \int_{\gamma_2} \varphi_k(\alpha_1) v_k h_1 d\alpha_1 + \int_Q F_k v_k dQ \tag{2.1}$$

(repeated indices indicate summation from 1 to 2),

$$\sigma_{12}|_{\gamma_2} = \varphi_1(\alpha_1), \quad \sigma_{22}|_{\gamma_2} = \varphi_2(\alpha_1), \quad \varphi_1, \varphi_2 \in L^2(\gamma_2), \quad F_1, F_2 \in L^2(Q).$$

LEMMA. For $\varepsilon > 0$ sufficiently small there exists a unique solution of the problem (2.1) and the following estimates, uniform with respect to ε:

$$\|u^\varepsilon\|_{V_0} \leqslant c, \quad \|e_{k2}(u^\varepsilon)\|_{L^2(Q)} \leqslant c, \quad k = 1, 2,$$
$$\varepsilon^{-2} \|e_{11}(u^\varepsilon)\|^2_{L^2} \leqslant c. \tag{2.2}$$

Since the proof of this lemma parallels that of Theorem 1 in [6], we shall not supply it here.

It follows from the Lemma that from the sequence u^ε we can select a subsequence (for which we retain the previous notation) such that $u^\varepsilon \to u^0$, weakly in V_0 and strongly in $L^2(Q)$, in accordance with an imbedding theorem due to Rellich. It follows from the last inequality in (2.2) that $e_{11}(u^\varepsilon) \to 0$, strongly in $L^2(Q)$, and, therefore, $e_{11}(u^0) = 0$, almost everywhere. We now put $K = \{v \in V_0;\ e_{11}(v) = 0\}$. It is obvious that K is a closed subspace of V_0. We examine two particular cases:

1) $\rho = [1/(h_1h_2)][(\partial h_1/\partial \alpha_2)] \neq 0$; 2) $\rho = 0$.

The function ρ has the geometric meaning of curvature of the family of curves $\alpha_2 \equiv$ const. In case 1

$$u_2^0 = -\frac{h_2}{h_{1,\alpha_2}} \frac{\partial u_1^0}{\partial \alpha_1} \in H^1(Q), \quad u_1^0 \in H^1(Q),$$

and we therefore have for u_1^0 the finite norm

$$\|u_1^0\|_L = \left\{\int_Q \left[\sum_{j,s=0}^{1} \left[\left(\frac{\partial^{1+j+s} u_1^0}{\partial \alpha_1^{j+1} \partial \alpha_2^s} \right)^2 + \left(\frac{\partial^{j+s} u_1^0}{\partial \alpha_1^j \partial \alpha_2^s} \right)^2 \right] \right] dQ \right\}^{1/2}. \tag{2.3}$$

According to a theorem of Banach concerning isomorphism, we can identify K with the space of functions L obtained upon completion of the functions v in the class $C^\infty(Q)$ with respect to the norm (2.3), such that $v|_{\gamma_1} = 0$, $\partial v/\partial \alpha_1|_{\gamma_1} = 0$. We now consider the integral identify (2.1) on functions $v \in K$ and we pass to the limit along the already chosen subsequence. We find that $u^0 = (u_1^0, u_2^0) \in K$ satisfied the identity

$$\int_Q [b_{22} e_{22}(u^0) e_{22}(v) + 4 e_{12}(u^0) e_{12}(v)]\, dQ = \int_{\gamma_2} \varphi_k(\alpha_1) v_k h_1 d\alpha_1 + \int_Q F_k v_k dQ \tag{2.4}$$

for all $v = (v_1, v_2) \in K$.

The limit problem consists in the determination of the functions u_1^0. The differential equation for u_1^0 is of the fourth order, of composite type with a double family of real characteristics $\alpha_2 \equiv$ const. The uniqueness of the solution of problem (2.4) is a consequence of Korn's second inequality: on V_0 we have the inequality

$$\int_Q \left[e_{11}^2(v) + b_{22}e_{22}^2(v) + 4e_{12}^2(v)\right] dQ \geqslant c\|v\|_{V_0}^2. \tag{2.5}$$

In the inequality (2.5) we put $v = u^0$ to obtain

$$a^0(u^0, u^0) \geqslant c\|u^0\|_{L^2}^2$$

where $a^0(u^0, v)$ is the bilinear symmetric form appearing in the left member of the equation (2.4). It follows from the uniqueness of u^0 that, in fact, the entire sequence u^ε converges weakly to u^0.

THEOREM 1. For $\varepsilon \to +0$ and $\rho \neq 0$ the solution of the problem (2.1) converges weakly in V_0 to the solution of the problem (2.4).

3. In Case 2 the subspace K is characterized by the condition $\partial v_1/\partial\alpha_1 = 0$. On γ_1 we have $v_1 = 0$ by hypothesis, and, from an inequality of Poincare type,

$$\int_Q v^2 dQ \leqslant c\int_Q \left(\frac{\partial v}{\partial \alpha_1}^2\right) dQ$$

it follows that $u_1^0 = 0$ almost everywhere in Q. Then

$$e_{22}(u^0) = \frac{1}{h_2}\frac{\partial u_2^0}{\partial \alpha_2},\ 2e_{12}(u^0) = \frac{h_2}{h_1}\frac{\partial}{\partial \alpha_1}\left(\frac{u_2^0}{h_2}\right).$$

Considering the integral identify (2.1) on K and then passing to the limit for $\varepsilon \to +0$, we find that u_2^0 is a solution of the variational problem

$$\int_Q [b_{22}e_{22}(u^0)\, e_{22}(v_2) + 4e_{12}(u^0)\ e_{12}\ (v)]\, dQ = \int_{\gamma_2} \varphi_2(\alpha_1)\, v_2 h_1 d\alpha_1 + \int_Q F_2 v_2 dQ \tag{3.1}$$

for every $v_2 \in H^1(Q)$, $v_2|_{\gamma_1} = 0$. The equation for u_2^0 is elliptic; the problem (3.1) has a unique solution for $\phi_2(\alpha) \in L^2(\gamma_2)$, $F_2 \in L^2(Q)$. From the uniqueness of the limit it follows that the initial sequence u^ε converges to u^0.

THEOREM 2. In Case 2 the solution of the problem (2.1) converges weakly in V_0 to the solution of the problem (3.1).

4. Let h_1, $h_2 = 1$; let the (α_1, α_2) coordinate system coincide with the orthogonal Cartesian (x_1, x_2) coordinate system; let Q be the planar rectangle $Q = \{(x_1, x_2);\ 0 \le x_1 \le 1,\ 0 \le x_2 \le h\}$. We take the boundary conditions for the mixed problem to be different from those in Sec. 1:

$$u_k^\varepsilon|_{x_2=0,h} = 0,\ \sigma_{1k}(u^\varepsilon)|_{x_1=0,1} = 0,\ k = 1, 2. \tag{4.1}$$

We now examine the limiting behavior as $\varepsilon \to +0$ of the problem (4.1). In this limiting process the estimates (2.2) are preserved; let $V_0 = |v = (v_1, v_2),\ v_k \in H^1(Q),\ v_k|_{x_2=0,h} = 0,$ $k = 1.2\}$. As we did before, we can select from the sequence u^ε a subsequence (for which we retain the previous notation) converging weakly in V_0 to the element $u^0 = (u_1^0, u_2^0) \in V_0$; moreover, $\partial u_1^0/\partial x_1 = 0$, $u_1^0 = u_1^0(x_2)$, $u_1^0(0) = u_1^0(h) = 0$ (in the weak sense). Let K be a subspace of V_0 characterized by the condition $\partial v_1/\partial x_1 = 0$. We consider the integral identity

$$\int_Q \sigma_{ij}(u^\varepsilon)\, e_{ij}(v)\, dx = \int_Q F_k v_k dx \tag{4.2}$$

on the subspace K, and we let $\varepsilon \to +0$. We find that u_1^0 and u_2^0 satisfy the integral identity

$$\int_Q \left[b_{22}\frac{\partial u_2^0}{\partial x_2}\frac{\partial v_2}{\partial x_2} + \left(\frac{\partial u_1^0}{\partial x_2} + \frac{\partial u_2^0}{\partial x_1}\right)\left(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1}\right)\right] dx = \int_Q F_k v_k dx \tag{4.3}$$

for every $v = (v_1, v_2) \in K$. The integral identity (4.3) can be written in the form

$$\int_Q \left[b_{22}\frac{\partial u_2^0}{\partial x_2}\frac{\partial v_2}{\partial x_2} + \frac{\partial u_2^0}{\partial x_1}\frac{\partial v_2}{\partial x_1}\right] dx + \int_0^h \frac{\partial u_1^0}{\partial x_2}\frac{\partial v_1}{\partial x_2} dx_2 + \int_0^h \frac{\partial u_1^0}{\partial x_2}(v_2(1, x_2) - \tag{4.4}$$

$$- v_2(0, x_2))\,dx_2 + \int_0^h \frac{\partial v_1}{\partial x_2}(u_2^0(1, x_2) - u_2^0(0, x_2))\,dx_2 = \int_0^h \widehat{F}_1(x_2)\,v_1 dx_2 +$$

$$+ \int_Q F_2 v_2 dx,\ \widehat{F}_1(x_2) = \int_0^1 F_1(x_1, x_2)\,dx_1.$$

Since $u_1^0 \in H^1(Q)$ and depends only on x_2, it follows that $u_1^0 \in H_0^1(0, h)$. The uniqueness of the solution of the limit problem is obvious (the difference of two solutions can only be a rigid displacement, which, in view of the homogeneous boundary conditions for $x_2 = 0$ and h, vanishes). The functions u_1^0 and u_2^0 satisfy, in the sense of distributions, the system of equations

$$b_{22}\frac{\partial^2 u_2^0}{\partial x_2^2} + \frac{\partial^2 u_2^0}{\partial x_1^2} = F_2,$$
$$\frac{\partial^2 u_1^0}{\partial x_2^2} - \frac{\partial}{\partial x_2}\left[u_2^0(1, x_2) - u_2^0(0, x_2)\right] = \widehat{F}_1 \qquad (4.5)$$

and the boundary conditions

$$u_k^0\big|_{x_2=0,h} = 0,\ k = 1, 2,\ \left.\frac{\partial u_1^0}{\partial x_2} + \frac{\partial u_2^0}{\partial x_1}\right|_{x_1=0,1} = 0. \qquad (4.6)$$

Moreover, $u_2^0 \in H^1(Q)$, $u_0^1 \in H_0^1(0, h)$.

THEOREM 3. As $\varepsilon \to +0$, the solution of the boundary value problem (4.1) converges weakly in V_0 to the solution of the boundary value problem (4.5), (4.6).

From the second of the equations (4.5) we can determine $u_1^0(x_2)$ as a function of $\hat{F}_1$ and u_2^0 and obtain for u_2^0 a nonlocal boundary value problem.

5. For media inextensible in the direction of the α_1-axis, the law relating the stresses and strains has the form

$$\sigma_{11} = q + b_{12}e_{22},\ \sigma_{22} = b_{22}e_{22},\ \sigma_{12} = 2e_{12},\ e_{11} = 0, \qquad (5.1)$$

where q is a new unknown function (Legendre multiplier) whose physical meaning has to do with the reaction of the medium to the presence of a kinematic constraint. As is the case in the solution of boundary value problems for the Navier–Stokes equations, q can be determined independently from the field of displacements. We write the system of equations corresponding to the law (5.1) in the differential form

$$L_1(u) = \frac{\partial}{\partial \alpha_1}(h_2 q) + P_1(u) = F_1 h_1 h_2, \qquad (5.2)$$
$$L_2(u) = -(q + b_{12}e_{22}(u))\frac{\partial h_1}{\partial \alpha_2} + P_2(u) = F_2 h_1 h_2,\ e_{11}(u) = 0.$$

Let K be a subspace of V_0 characterized by the condition $e_{11}(v) = 0$. If the first equation of the system (5.2) is multiplied by v_1, the second by v_2, and if we add the resulting equations and make an integration by parts, we find that u_1 and u_2 satisfy the integral identity (2.4), the term with q dropping out due to the fact that $(q, e_{11}(u))_{L^2} = 0$. Conversely, if it is known, with respect to a generalized solution of the system (5.2), that it is sufficiently smooth, we can transform the identity (2.4) to the form

$$\int_Q [(P_1(u) - F_1)\,v_1 + (P_2(u) - F_2)\,v_2]\,dQ = 0$$

for $v \in K$. But it then follows from the results given in [7] (Corollary 4.1) that there exists a function $q \in L^2(Q)$ such that, in the sense of distributions, the system of equations (5.2) is valid.

THEOREM 4. As $\varepsilon \to +0$, the solution of the problem (2.1) converges strongly to u^0 in V_0; moreover, $\|u^\varepsilon - u^0\|_{V_0} \le c\varepsilon$.

We assume, for simplicity, that the boundary conditions on γ_2 are homogeneous. Then

$$a^\varepsilon(u^\varepsilon - u^0, v) = \int_Q \varepsilon^{-2} e_{11}(u^\varepsilon) e_{11}(v)\, dQ + \int_Q b_{22} e_{22}(u^\varepsilon - u^0) e_{22}(v)\, dQ + \int_Q 4 e_{12}(u^\varepsilon - u^0) e_{12}(v)\, dQ = -\int_Q q e_{11}(v)\, dQ. \quad (5.3)$$

In Eq. (5.3) we put $v = u^\varepsilon - u^0$ and we estimate the right side as follows:

$$\left| \int_Q q e_{11}(u^\varepsilon - u^0)\, dQ \right| \leqslant \|q\|_{L^2} \|e_{11}(u^\varepsilon)\|_{L^2} \leqslant \frac{1}{2\varepsilon^2} \|e_{11}(u^\varepsilon)\|_{L^2}^2 + \frac{\varepsilon^2}{2} \|q\|_{L^2}^2.$$

Therefore

$$c_1 \|u^\varepsilon - u^0\|_{V_0}^2 \leqslant \frac{\varepsilon^{-2}}{2} \int_Q e_{11}^2(u^\varepsilon)\, dQ + \int_Q b_{22} e_{22}^2(u^\varepsilon - u^0)\, dQ + \int_Q 4 e_{12}^2(u^\varepsilon - u^0)\, dQ \leqslant \frac{\varepsilon^2}{2} \|q\|_{L^2}^2.$$

It follows from this that

$$\|u^\varepsilon - u^0\|_{V_0} \leqslant c_2 \varepsilon, \quad \|\varepsilon^{-2} e_{11}(u^\varepsilon)\|_{L^2}^2 \leqslant c_3 \quad (5.4)$$

and the constants c_2 and c_3 do not depend on ε. It follows from the second of the inequalities (5.4) that $\varepsilon^{-2} e_{11}(u^\varepsilon)$ converges to the function weakly in $L^2(Q)$. Apparently, q and q_0 are coincident.

6. In the above we have assumed that the volume forces did not depend on ε. Then, in the limit the deformation $e_{11}(u^0) = 0$. However, a situation is possible in which $e_{11}(u^0) \neq 0$. Let Q be a bounded domain on the plane, let the system of orthogonal coordinates be Cartesian, and on the boundary of the domain, let us specify zero displacements. The variational problem is formulated in the following way: Determine a two-component function $u^\varepsilon \in [H_0^1(Q)]^2$, satisfying for every $v = (v_1, v_2) \in [H_0^1(Q)]^2$ the integral identity

$$a^\varepsilon(u^\varepsilon, v) = \int_Q [F_1 \varepsilon^{-2} v_1 + F_2 v_2]\, dx, \quad F_1, F_2 \in L^2(Q). \quad (6.1)$$

A similar kind of dependence of the right side on ε can be obtained if we assume that initially we have nonzero displacements specified on the boundary. In reducing our problem to a homogeneous one, we obtain the integral identity (6.1). In this case the norm of u^ε is, generally speaking, unbounded in $[H_0^1(Q)]^2$. Let us examine the limiting behavior of the problem (6.1) as $\varepsilon \to +0$.

We introduce the Hilbert space W of functions obtained upon completion of functions of the class $C_0^\infty(Q)$ in the norm

$$\|u\|_W = \left\{ \int_Q \left[u^2 + \left(\frac{\partial u}{\partial x_1} \right)^2 \right] dx \right\}^{1/2}.$$

The following Poincaré type inequality holds on functions from W:

$$\int_Q u^2 dx \leqslant c \int_Q \left[\frac{\partial u}{\partial x_1} \right]^2 dx.$$

We can show that functions from W assume a zero value in the mean on an arbitrary connected component of the boundary having no horizontal tangent.

<u>THEOREM 5.</u> As $\varepsilon \to +0$, u_1^ε converges weakly in W to the solution of the equation

$$\frac{\partial^2 u_1^0}{\partial x_1^2} = F_1, \quad F_1 \in L^2(Q), \quad u_1^0 \in W,$$

and u_2^ε converges weakly in $H_0^1(Q)$ to the solution of the equation

$$b_{22} \frac{\partial^2 u_2^0}{\partial x_2^2} + \frac{\partial^2 u_2^0}{\partial x_1^2} = F_2 - (1 + b_{12}) \frac{\partial^2 u_1^0}{\partial x_1 \partial x_2}, \quad u_2^0 \in H_0^1(Q). \quad (6.2)$$

We note that the right side of equation (6.2) belongs to $H^{-1}(Q)$. The proof of Theorem 5 is analogous to the proof of Theorem 3 in [8]; it is therefore omitted here.

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CONTACT PROBLEM OF THE THEORY OF ELASTICITY FOR PRESTRESSED BODIES WITH CRACKS

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Fatigue-test results often have a large scatter, which is generally related to a range of uncontrollable factors including the structure of an residual-stress distribution in the surface layers of the material, errors in assembly of the part, instability of the regime parameters and lubricant properties, etc. The effect of some of these factors on the performance of machine parts such as bearings was examined in [1]. There has been less study of the effect of residual stresses unavoidably created by some type of treatment (thermal, thermochemical, mechanical work-hardening, etc.) on the contact fatigue of materials. This topic has been investigated only by experimental method, and the available literature sources do not offer an unambiguous treatment of this subject. For example, in [2] (p. 227), the authors dispute that residual stresses have a significant effect on the fatigue of bearing steels. Several authors [3-7] hold that the retardation of fatigue is favorably influenced by compressive residual stresses and unfavorably influenced by tensile residual stresses. Other studies [8] indicate that compressive stresses are intolerable and that small tensile residual stresses are useful. Thus, the question of the usefulness and measurement of the effect of residual stresses on fatigue fracture remains unanswered.

Experimental studies were made in [9, 10] on the effect of shear stresses on contact fatigue. It was found that such stresses have an adverse effect on the fracture process.

Here we propose a mechanical model for the combined effect of normal and shearing contact stresses on fracture on the one hand and, on the other hand, the effect of residual stresses in the surface layers on fracture. The problem is examined in an elastic formulation and is reduced to a system of integral and integrodifferential equations with additional conditions in the form of equalities and inequalities. A solution is obtained by asymptotic methods. We determine the distribution of contact stresses and the stress intensity factors at the crack tips. An analysis is made of the effect of different levels of shearing contact stresses and residual stresses, as well as their sign (tensile or compressive), on the stress intensity factors. Numerical results are presented.

Thus, on the basis of analysis of the proposed model, it is possible to comparatively evaluate the effect of the above-mentioned factors on contact fatigue.

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